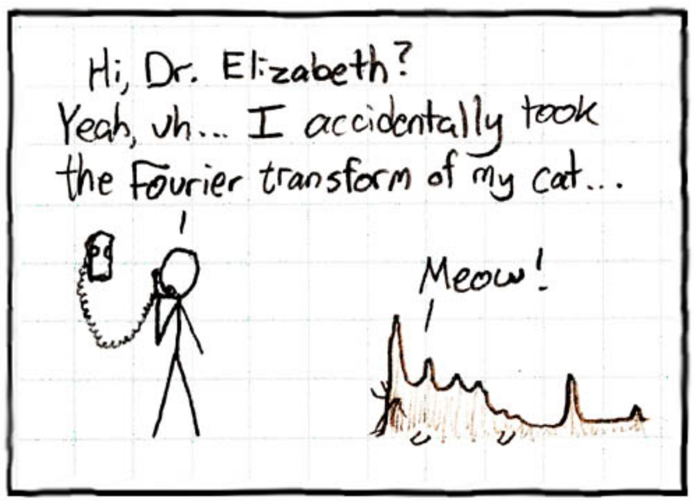
Fourier Transforms and Image Formation

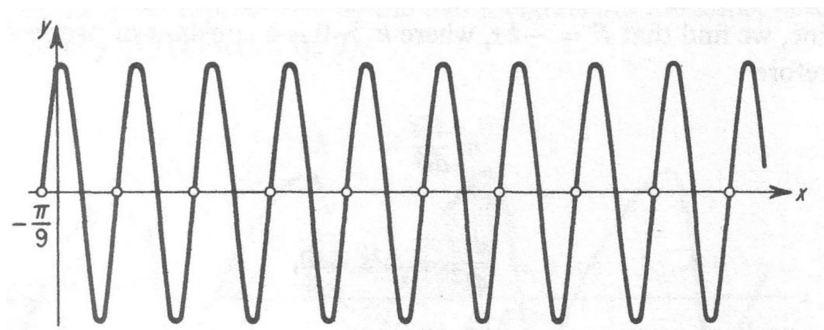


Essential Concepts

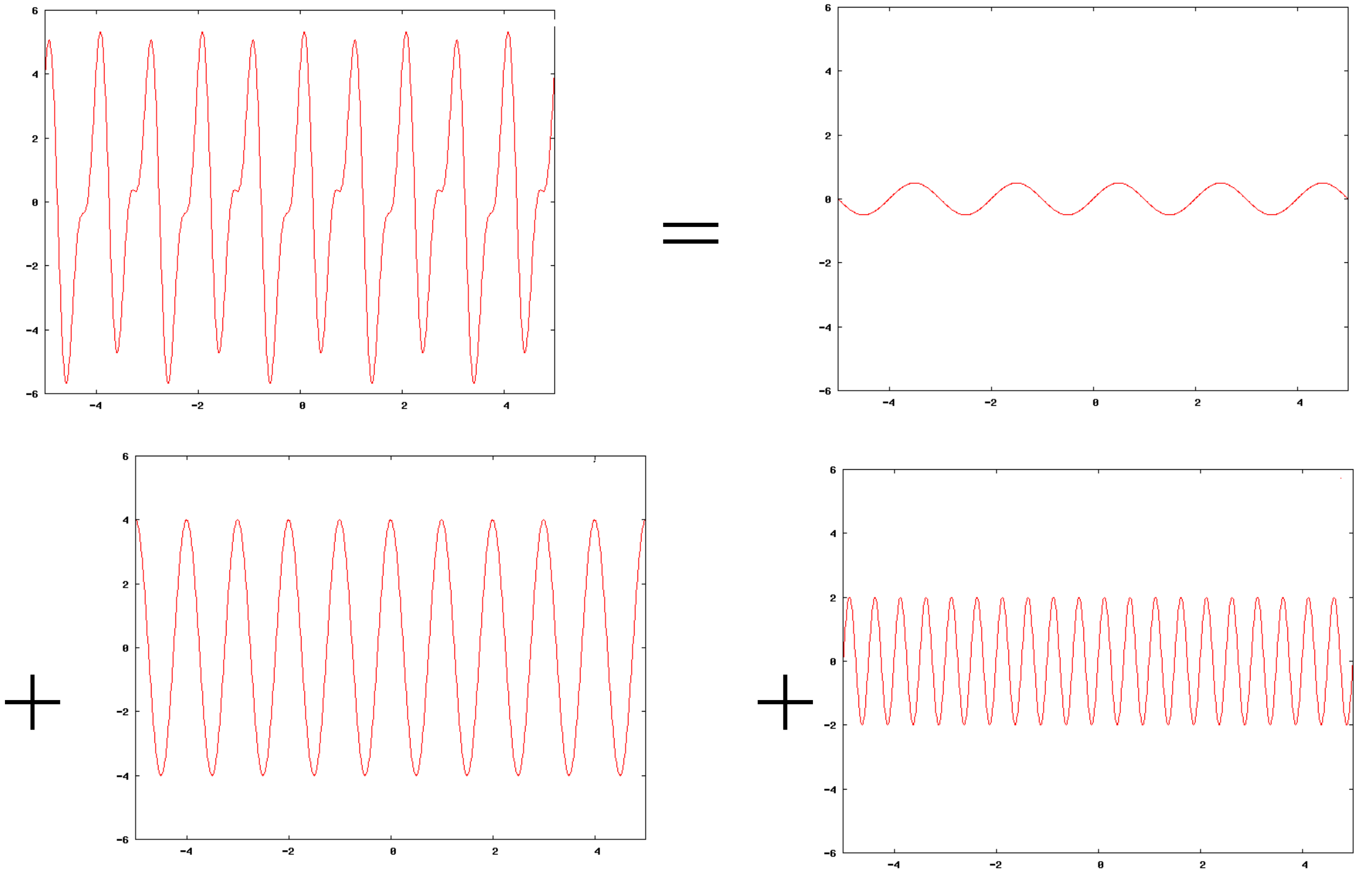
- Information from viewing FFT
- Four essential 1D Fourier transform mates
- Convolution theorem, correlation
- Sampling and Nyquist limit
- Projection theorem

Sine Wave

- $y = 2\sin(3x + \frac{\pi}{3})$

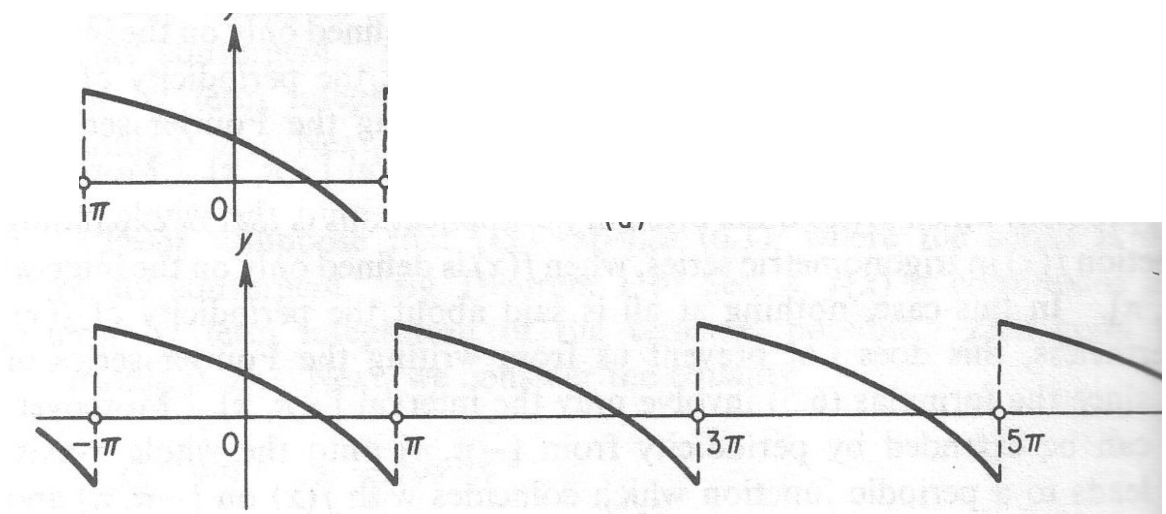


Fourier Series



Fourier Series

- Non-periodic Functions
- For functions which are defined only on the interval [-π,π], we extend the function by periodicity onto the whole x-axis:



Conventions

- **Image domain**
- Real space
- f(x,y)
- f(r,θ)
- f(t)

- **Fourier Domain**
- Fourier Space
- Inverse space
- Reciprocal space
- Diffraction space
- F(X,Y), F(kx,ky)
- F(k,Θ)
- H(ω)

Important Transform Pairs

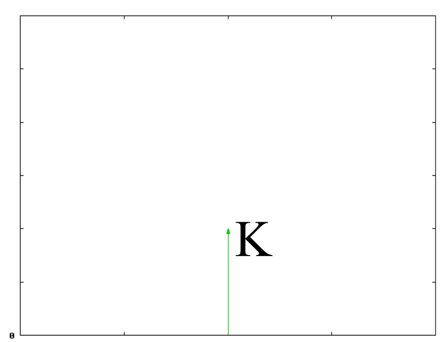
- Delta
- Top-hat
- Comb
- Gaussian

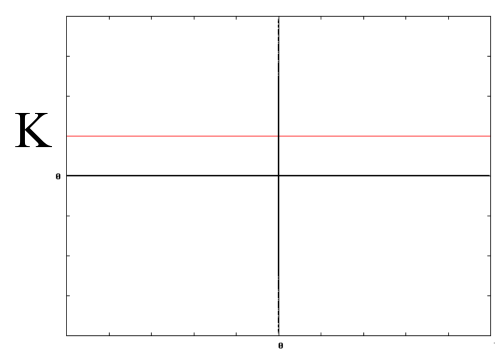
Fourier transform of a delta function

$$h(t) = K\delta(t)$$

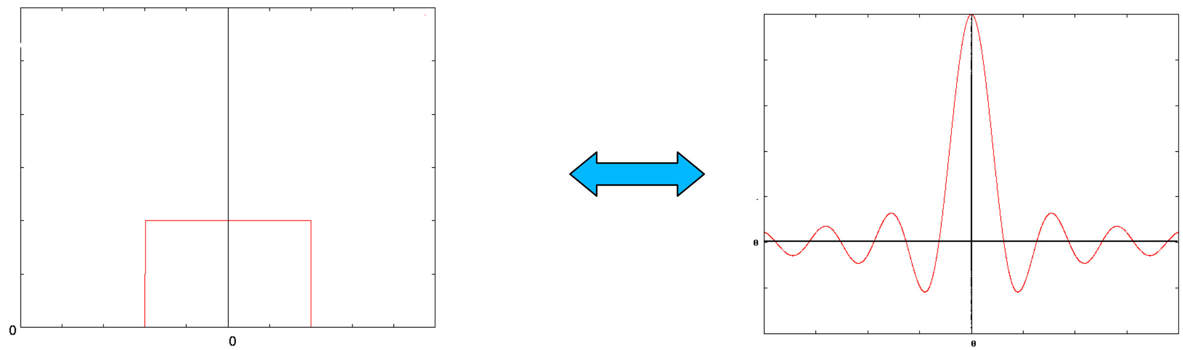
$$H(f) = \int_{-\infty}^{\infty} K\delta(t)e^{-2\pi ift}\,dt$$

$$H(f) = K$$





Top-hat function

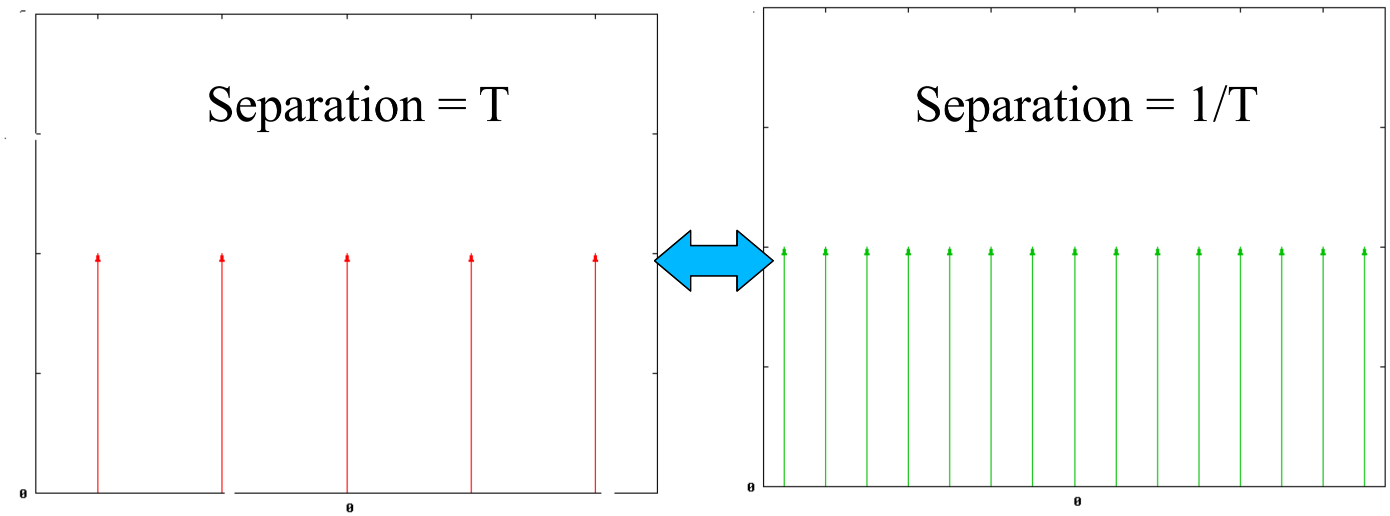


$$H(X) = \frac{\sin \pi X}{\pi X}$$

sinc function

Comb function

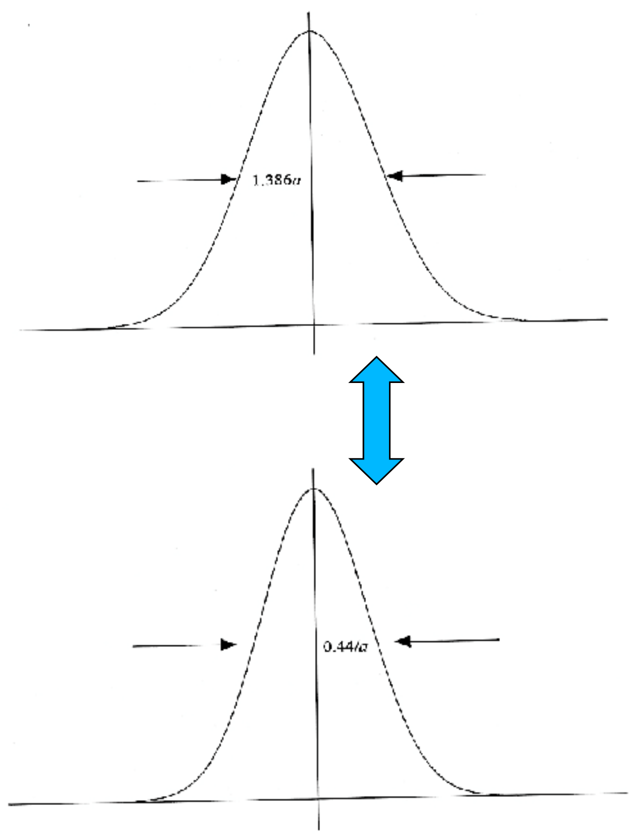
- Infinite points on comb



$$h_\infty(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$H_\infty(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$$

Gaussian Function



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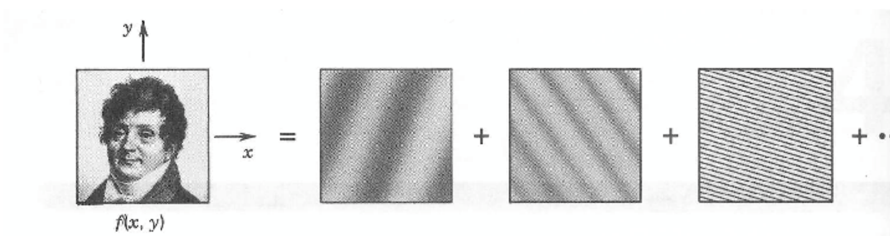
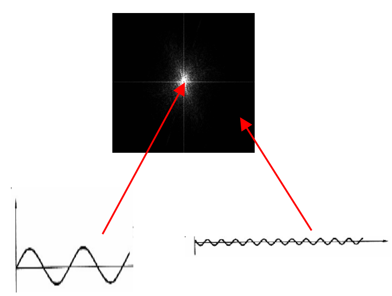
Images: Extending transforms into 2d

- Consider images as a 2 dimensional function of x and y, where the value of f(x,y) is represented as brightness

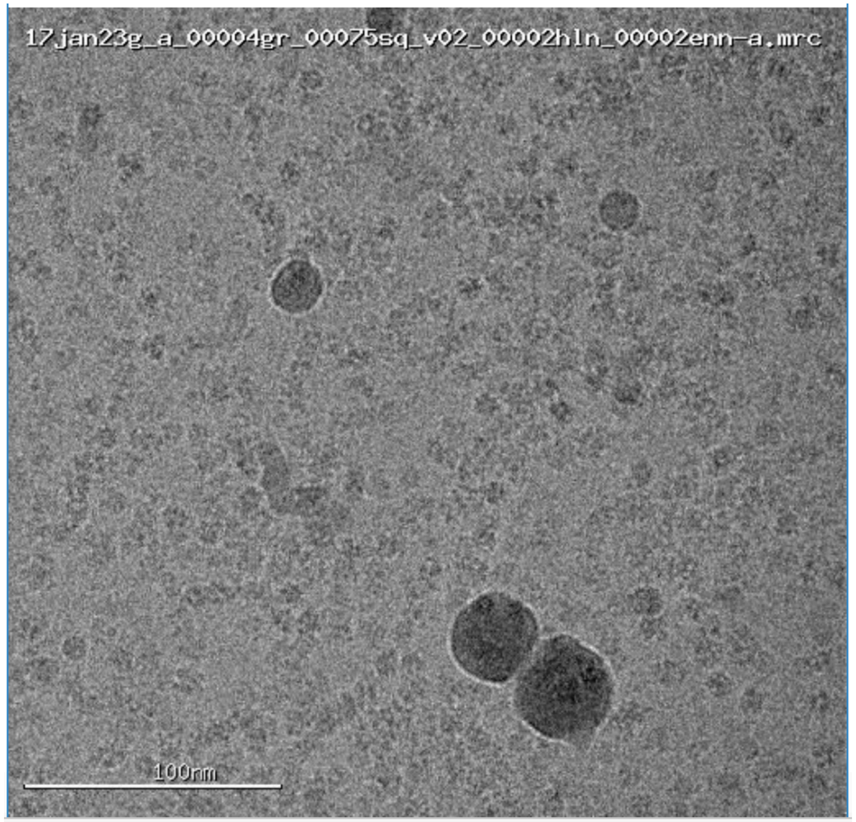
2D Fourier transform

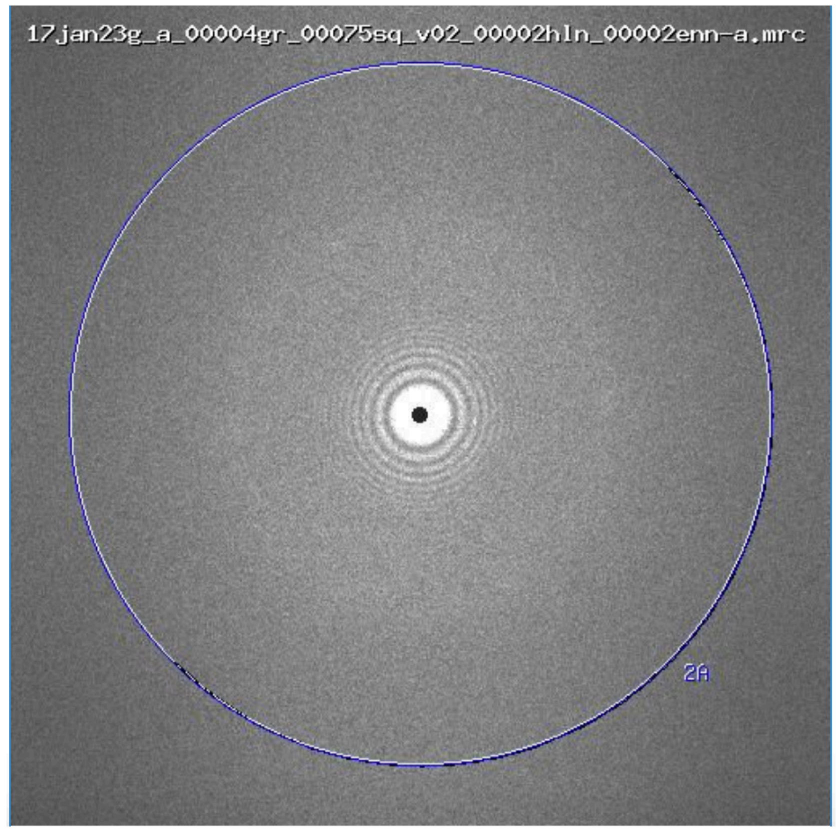




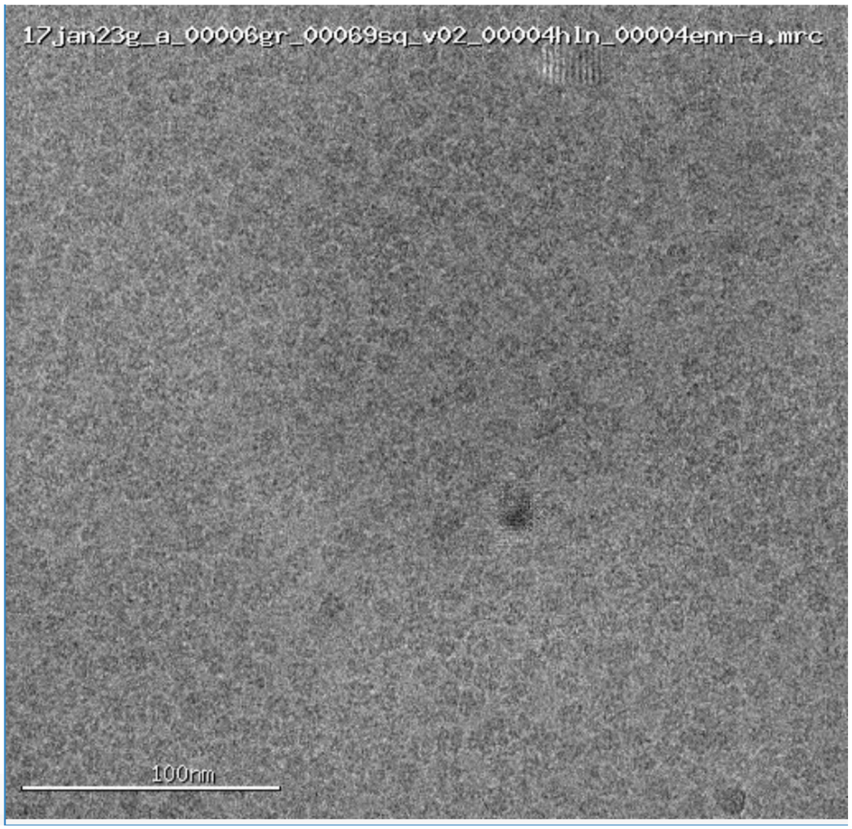
$$F(K_x, K_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\exp(i(K_x x + K_y y))dxdy$$

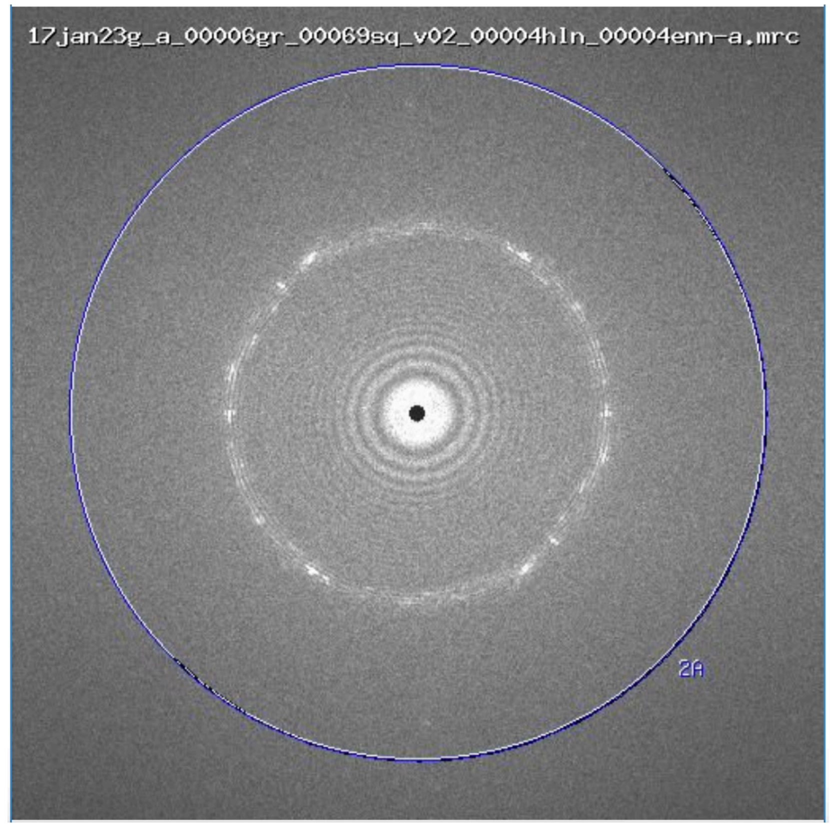
Information from visualizing FFT



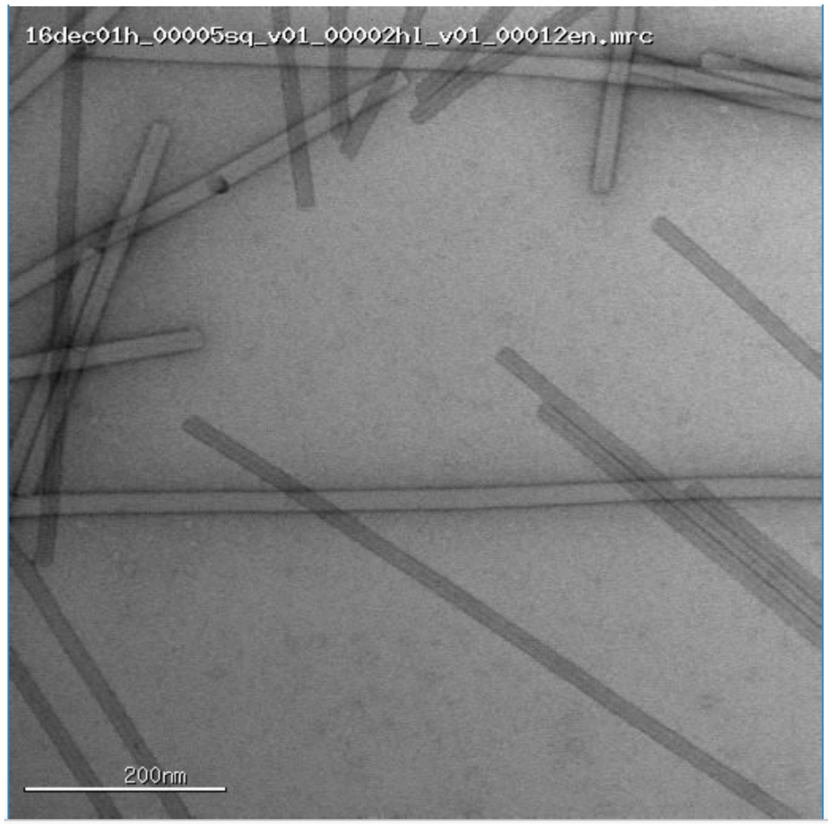


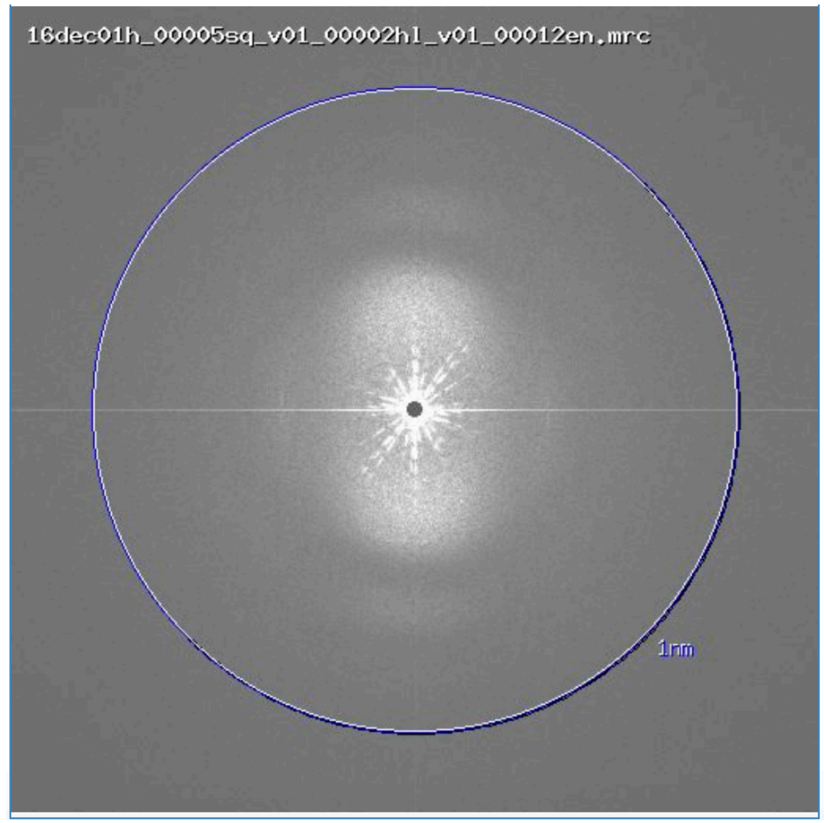
Information from visualizing FFT





Information from visualizing FFT





Importance of Phase and Amplitude

1 

Calculate FFT Keep amplitude

2 

Calculate FFT Keep phase

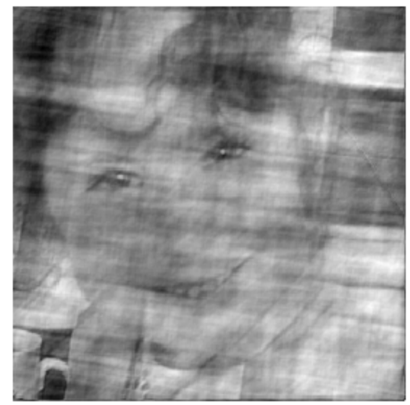
Importance of Phase and Amplitude

1

2

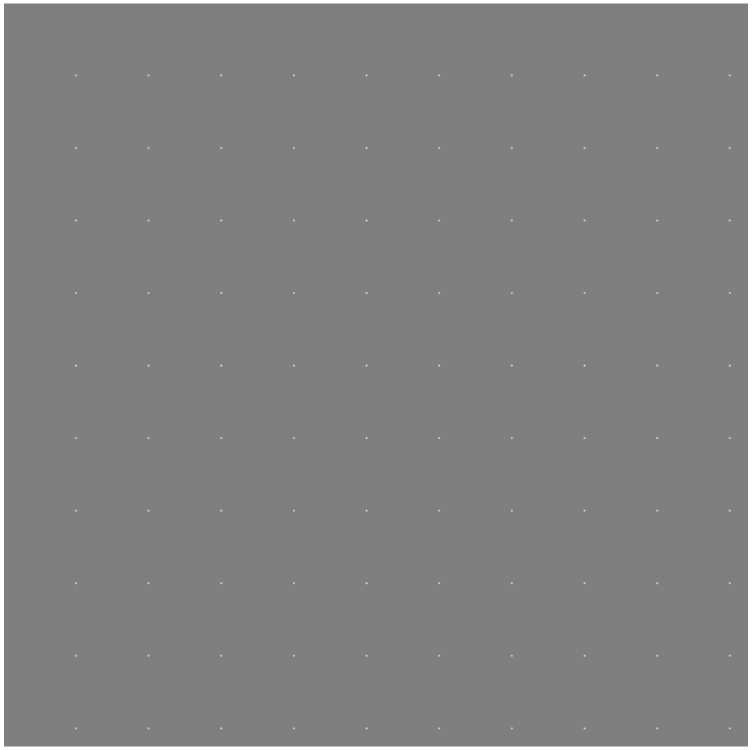
Amplitude of object 1
Phase of object 2

Inverse FT



Reconstituted image is dominated by phase

Convolution with set of delta functions



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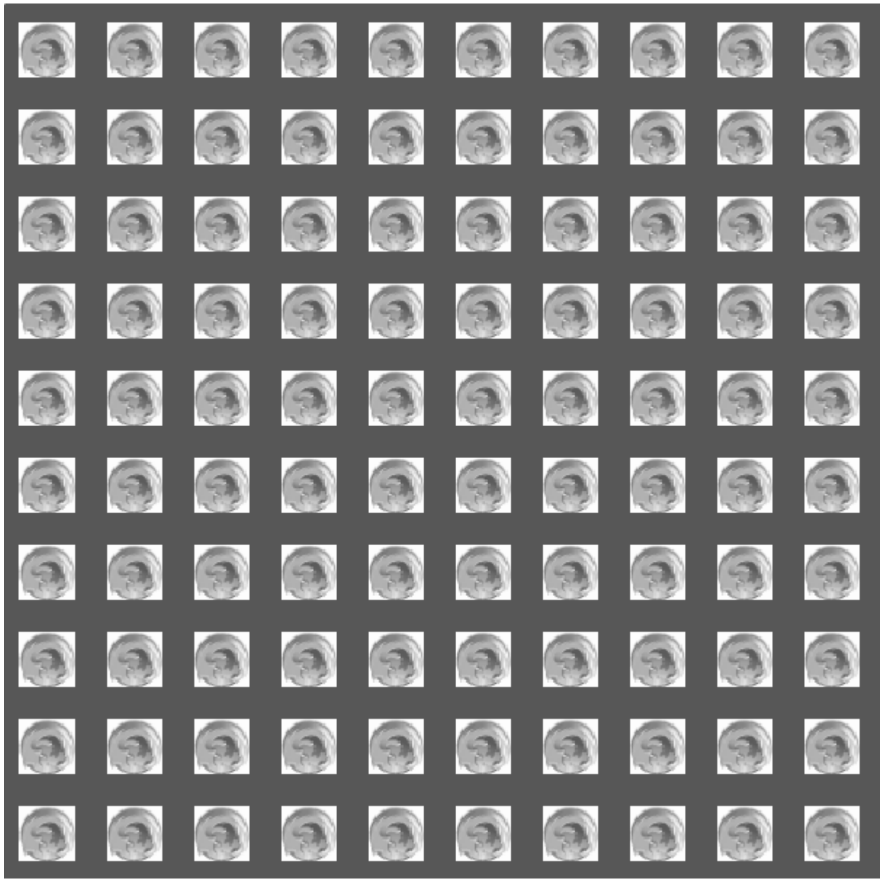
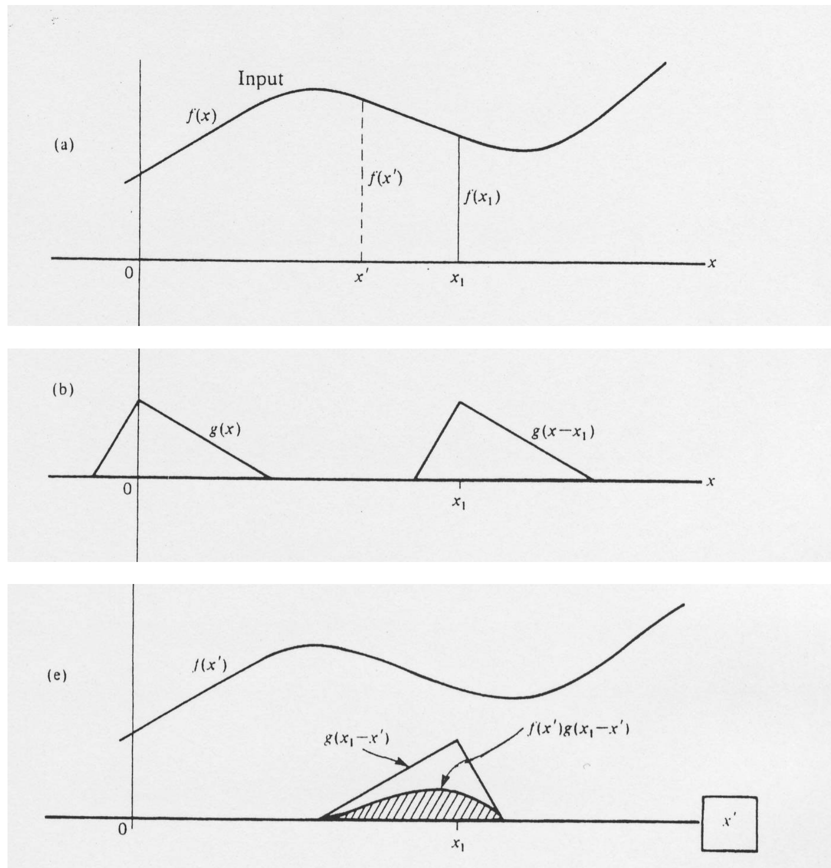
Convolution with set of delta functions



Convolution

Say we have a function f(x) input to a device

and another function g(x) scanning function

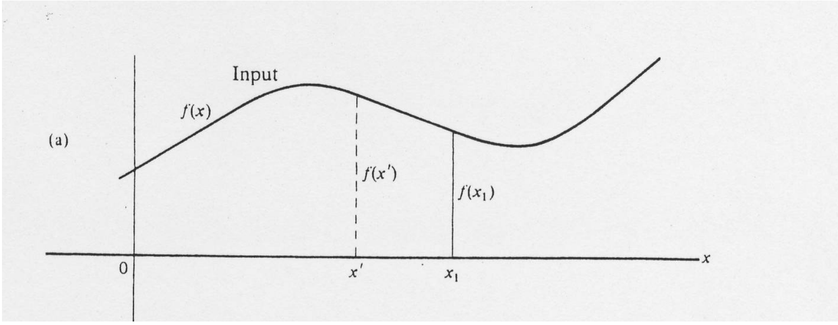


Slide the function g(x) along the curve of f(x)

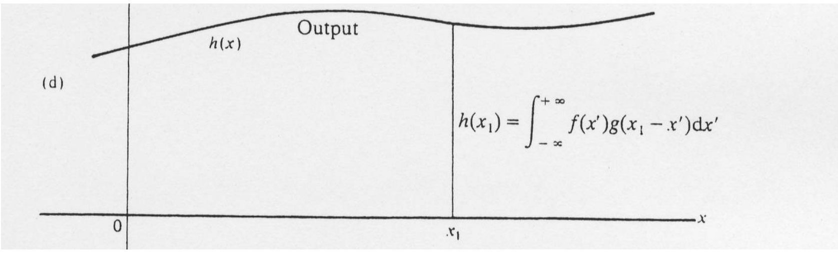
g(x) is imperfect, and “smears” f(x)

Point at x1 is replaced by value of shaded integral

Convolution



Input



resultant output

Convolution Theorem

if $h(x)$=f(x)*g(x)

then $H(X) = F(X)\times G(X)$

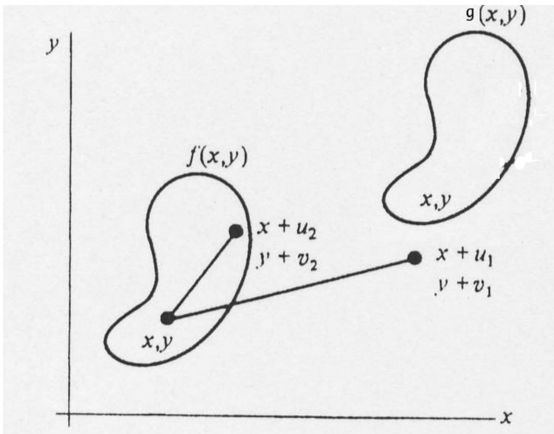
Conversely,

If $H(X) = F(X) * G(X)$

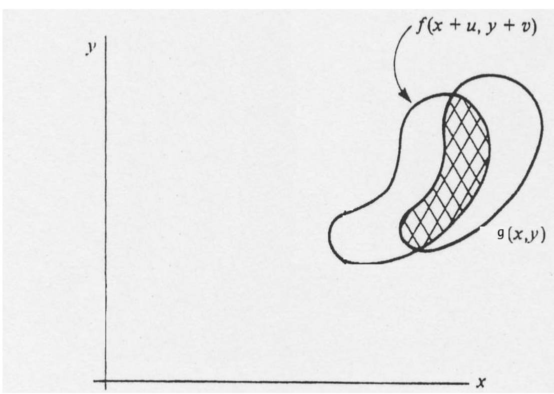
$h(x) = f(x) \times g(x)$

Convolution is easy to calculate:
take FT of each function, multiply, then take inverse FT

Correlation



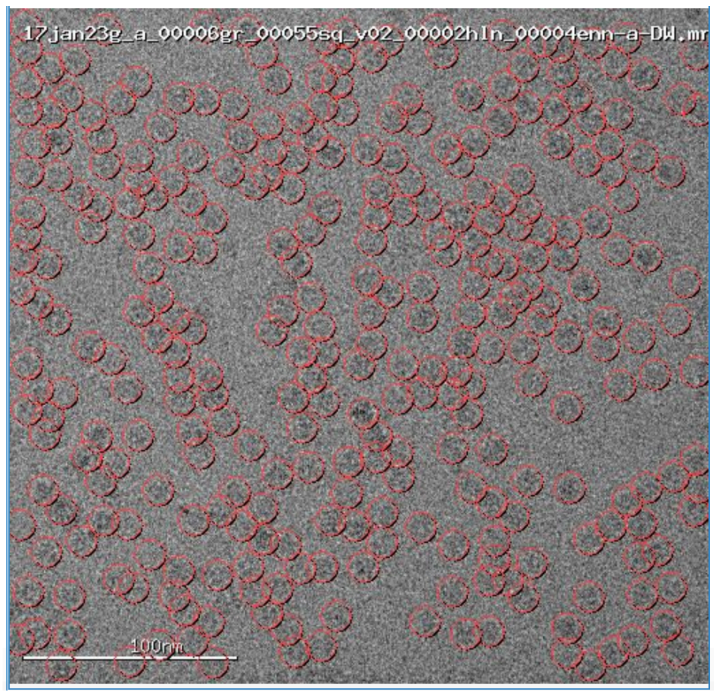
"slide" f(x,y) over g(x,y) and calculate overlap

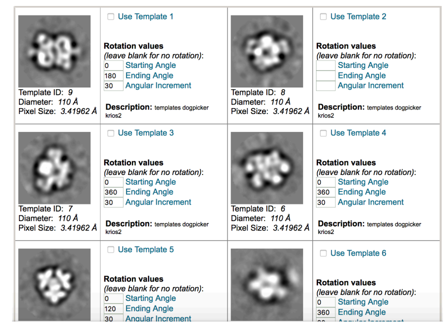


formally, calculate a function P

$$P(x,y) = \iint_{-\infty}^{\infty} f(x',y')g(x'-x,y'-y)dx'dy'$$

Example: Particle Picking





Correlation and Convolution

$$h(x) = \int_{-\infty}^{\infty} f(x')g(x-x')dx$$ Convolution

$$\text{p}(x) = \int_{-\infty}^{\infty} f(x')g(x'-x)dx$$ Correlation

Convolution Theorem

$h(x)=FT^{-1}\{F(X)G(X)\}$

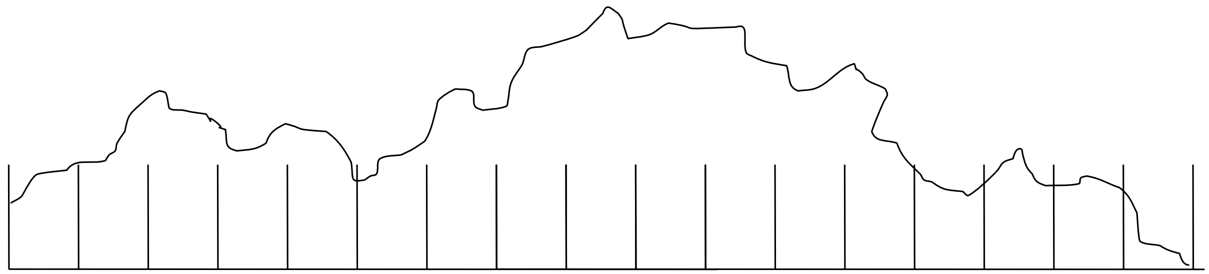
$p(x)=FT^{-1}\{F(X)G^*(X)\}$

$G^*(X)$: complex conjugate of $G(X)$

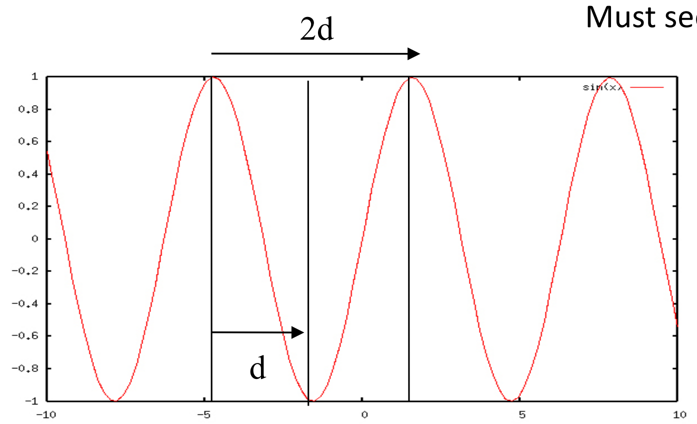
Complex conjugate:
if $x=a+ib$, $x^* = a-ib$

Image sampling (for digital FT)

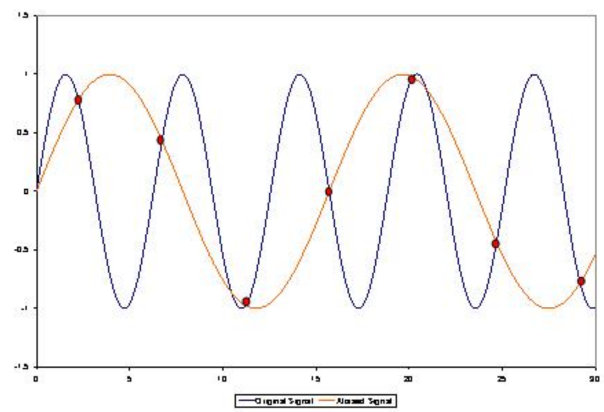
Shannon-Nyquist sampling limit:
Finest spatial period must be sampled >2x
Otherwise → aliasing



Must see peaks and valleys of a feature

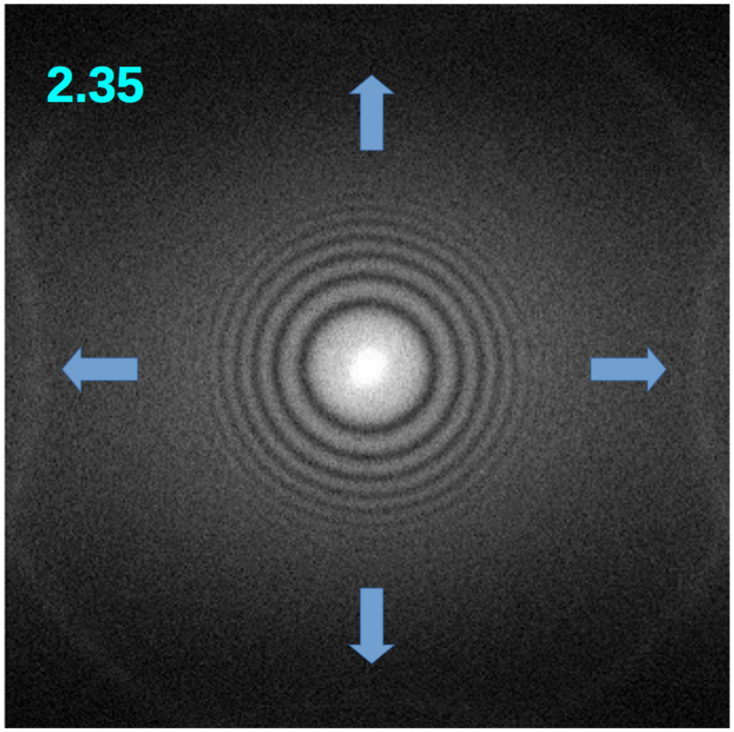


A minimally sampled image



Undersampled

30K: Gold Aliases Back into Image

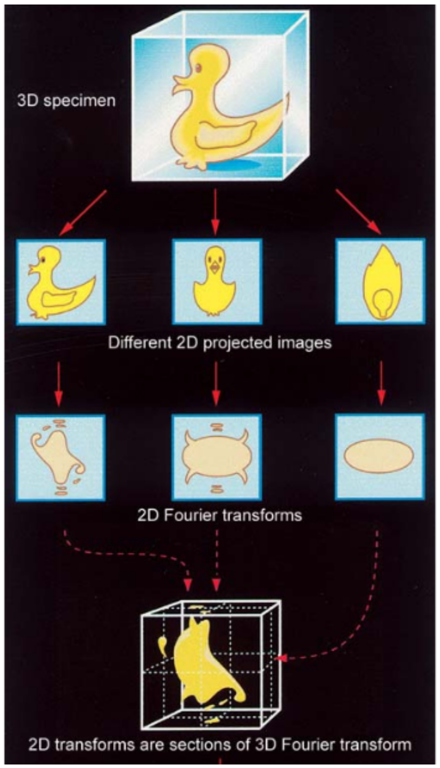


Pixel Size: 1.2 A

Aliasing

- By sampling our function (image) at scanning frequency s, we are essentially multiplying by a set of delta functions with spacing s.
- Therefore in Fourier space, we are convoluting with a set of delta functions with spacing 1/s

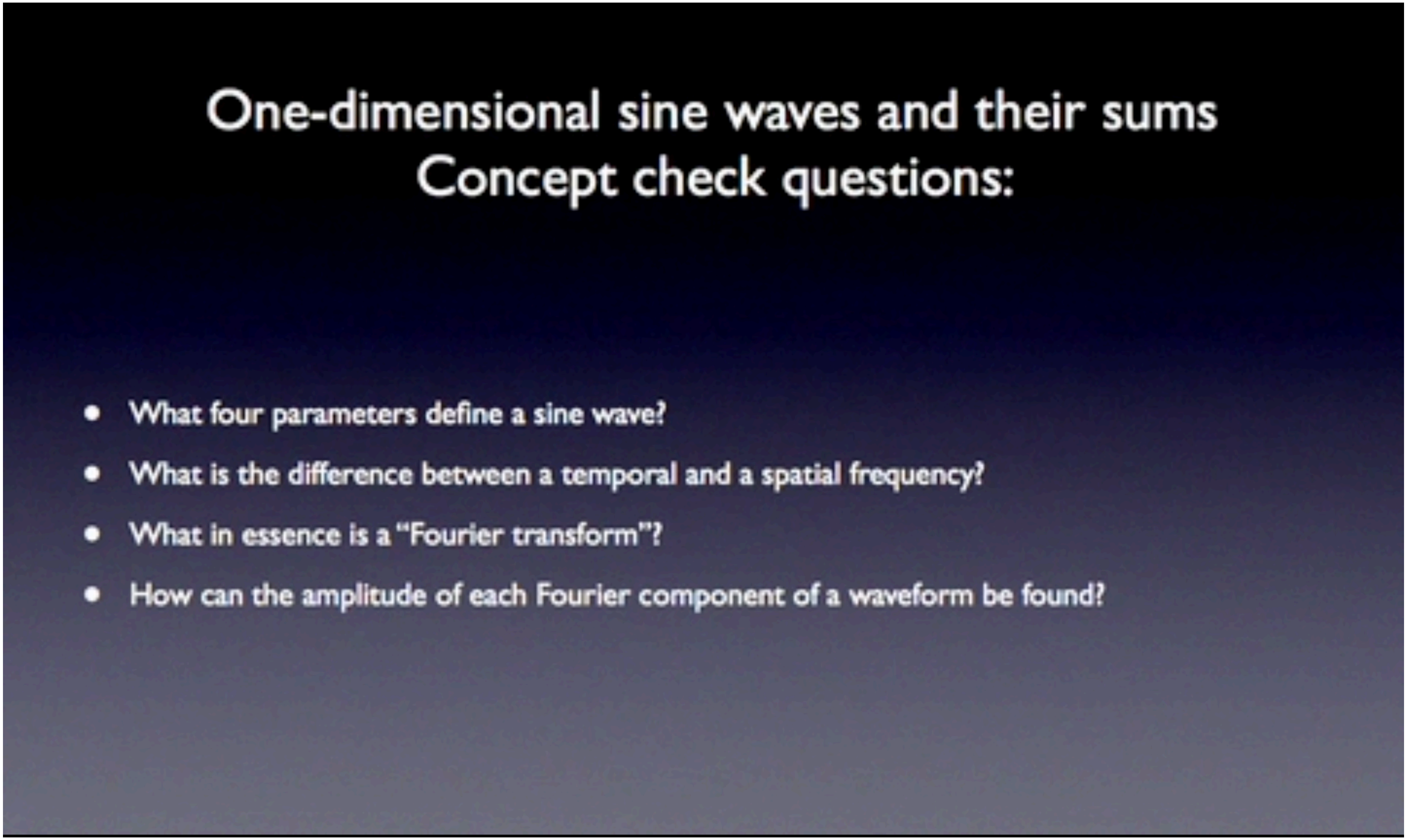
Projection Theorem
(Central Slice Theorem)



From Baker and Henderson (2012), International Tables of Crystallography Vol. F, Ch. 19.6, pp. 593-614.

Fourier Inversion

Questions


One-dimensional sine waves and their sums
Concept check questions:
What four parameters define a sine wave?
What is the difference between a temporal and a spatial frequency?
What in essence is a "Fourier transform"?
How can the amplitude of each Fourier component of a waveform be found?

One-dimensional reciprocal space
Concept check questions:

- What is the difference between an “analog” and a “digital” image?
- What is the “fundamental” frequency? A “harmonic”? “Nyquist” frequency?
- What is “reciprocal” space? What are the axes?
- What does a plot of the Fourier transform of a function in reciprocal space tell you?

19:39 -00:28

Two-dimensional waves and images
Concept check questions:

- What does a two-dimensional sine wave look like?
- What do the "Miller" indices "h" and "k" indicate?

Two-dimensional transforms and filters
Concept check questions:

- In the Fourier transform of a real image, how much of reciprocal space (positive and negative values of "h" and "k") is unique?
- If an image "I" is the sum of several component images, what is the relationship of its Fourier transform to the Fourier transforms of the component images?
- What part of a Fourier transform is not displayed in a power spectrum?
- What does the "resolution" of a particular pixel in reciprocal space refer to?
- What is a "low pass" filter? "High pass"? "Band pass"?

Convolution and cross-correlation
Concept check questions:

- What is a "convolution"?
- What is the "convolution theorem"?
- What is a "point spread function"?
- What does convolution have to do with the structure of crystals?
- What is "cross-correlation"?
- How might cross-correlations be used in cryo-EM?

Details

Fourier Series

- Final Fourier series expansion (after removing 2 π):
- $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k coskx + b_k sinkx)$

Terms a_k and b_k:

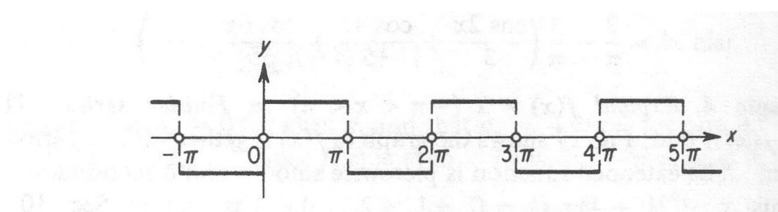
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx$$

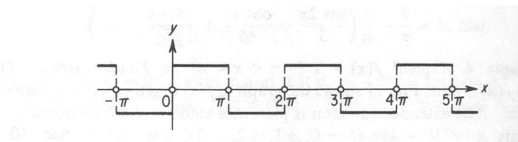
Fourier Series

- Example: Determine the terms for the Fourier series expansion of
- $f(x)=-1,\ -\pi<x<0$
- $f(x)=0,\ x=(0,-\pi,\pi)$
- $f(x)=1,\ 0<x<\pi$

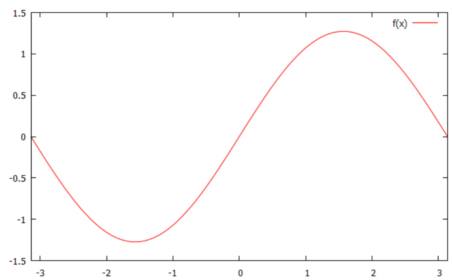
Reference:
Fourier Series by Georgi P Tolstov
Translated by Richard A Silverman
Dover Publications Inc :
New York (1972)



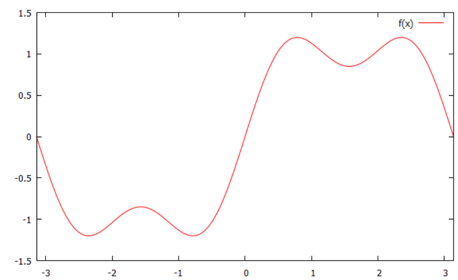
Fourier Series



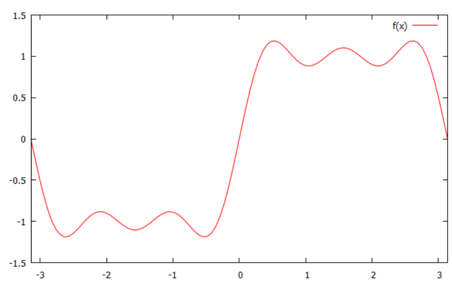
$$(f(x) = \frac{4}{\pi}(\sin x + \ \sin\frac{3x}{3} + \ \sin\frac{5x}{5} + \ \ldots)$$



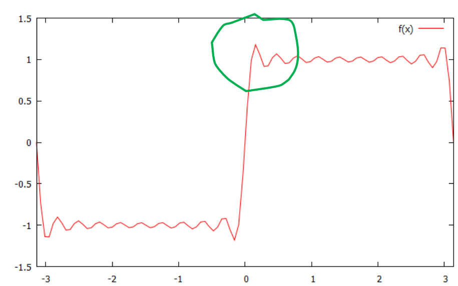
1 term



2 terms



3 terms



10 terms

Fourier Series: complex form

- From the Euler equation
- $e^{i\varphi} = cos\varphi + i\ sin\varphi$

- Can derive
- $\cos kx = \frac{e^{ikx}+e^{-ikx}}{2}$
- $\sin kx = i\,\frac{-e^{ikx}+e^{-ikx}}{2}$

Fourier Transforms

- Take the limit of the series as period l→∞: (and much manipulation)

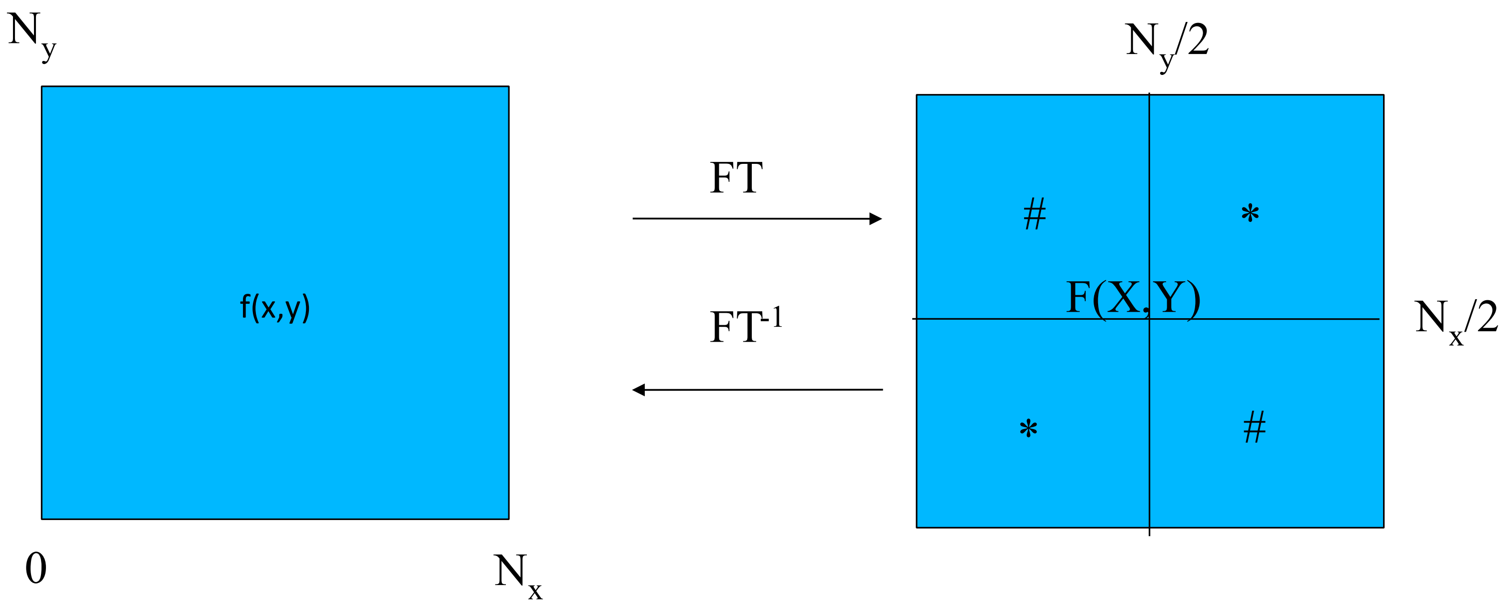
- $F(X) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i X x} dx$
- The inverse transform is:
- $f(x) = \int_{-\infty}^{\infty} F(X) e^{2\pi i x X} dX$

- The two equations are mates, and let you convert from real space to frequency space and back

Fourier Transforms of Images

- Computationally, images are a discrete matrix of points, and so the computer actually calculates the discrete Fourier transform (DFT)
- We are back to sums instead of integrals
- The Fast Fourier transform (FFT) is efficient: on order n log(n) rather than n^2 (Cooley and Tukey, 1965; Gauss 1805)
- Originally for n of power of 2, it can be calculated for all n, though power of 2 is simpler
- Generally only the amplitude is displayed

Fourier transform is an invertible operator

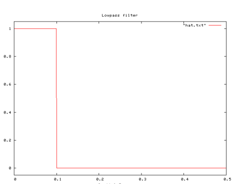


*, # - Friedel mates

Fourier Filters

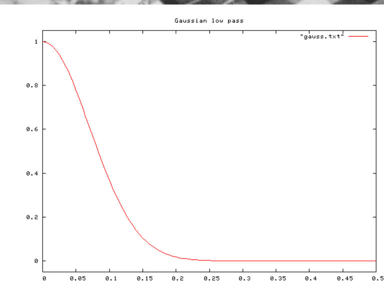
- Low pass filters
  - Restrict data to low frequency components
  - blurs out features (filter out high frequency noise)
- High pass filters
  - Restrict data to high frequency components
  - Eliminate gradients in images
- Band pass filters
  - Restrict data to a band of frequencies
- Band stop filters
  - suppress a band of frequencies

Low-pass filter



“ringing” effects caused by sharp mask

Gaussian Low-pass filter



Continuous or smoother mask prevents ringing

$$e^{\frac{X^2}{2 \times RAD^2}}$$

Convolution

- If we have 2 functions f(x) and g(x)
- The **convolution** of f(x) with g(x) is defined as
- $h(x) = \int_{-\infty}^{\infty} f(x')g(x - x')dx'$
- $h(x) = f(x) * g(x)$
- Reverse one function, offset it by x’ (starting at -∞), multiply by second function, and measure area under resultant curve. Slide from -∞ to ∞.

Further Reading

- Atlas of Optical Transforms by G. Harburn, C.A. Taylor, T.R. Welberry. Cornell University Press: Ithica, New York (1975)
- The Fourier Transform and its Applications, 2nd Edition by Ronald L. Bracewell. Mcgraw Hill Book Company: New York (1978)
- Fourier Optics: An Introduction (2nd Edition) by E.G. Steward. Dover Publications Inc.: Mineola, New York (2004)
- Fourier Series by Georgi P Tolstov (Translated by Richard A Silverman). Dover Publications Inc : New York (1972)